VI. *On the Calculus of Symbols.*

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IN a paper published in the Philosophical Transactions for 1861, p. 79, Mr. W. H. L. RUSSELL has constructed systems of multiplication and division for functions of non-commutative symbols, subject to the same laws of combination as those in Professor BOOLE'S memoir "On a General Method in Analysis," Philosophical Transactions, 1844. In this calculus there are four systems of multiplication and division, viz. internal and external, both (1) when the functions are arranged in powers of ϱ, (2) when in powers of π, or, as they may be termed, the ϱ-arrangement, and the π-arrangement. In the paper in question the author has confined himself, so far as division is concerned, to the case most useful in practice, in which the divisor is linear. And of this he has discussed in full only the ϱ-arrangement.

§ 1. *Internal division of the π-arrangement by a linear factor.*

Adopting the same notation as Mr. RUSSELL, I propose here to investigate, in the first place, the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an *internal* factor of

$$\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\ldots\varphi_0(\varrho),$$

and to determine the quotient. This is partially discussed in pp. 73–75.

Let

$$\varrho\frac{d}{d\varrho}\psi=\psi',$$

then performing the actual divisions, for brevity writing ψ for $\psi(\varrho)$, and φ for $\varphi(\varrho)$,

$$\begin{array}{r|l} \psi_1\pi+\psi_0\Big) & \varphi_1\pi+\varphi_0\Big(\dfrac{\varphi_1}{\psi_1} \\ & \underline{\varphi_1\pi+\varphi_1\dfrac{\psi_0}{\psi_1}} \\ & \quad\varphi_0-\varphi_1\dfrac{\psi_0}{\psi_1}. \end{array}$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0}{\psi_1}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Again,

$$
\begin{array}{l}
\psi_1\pi+\psi_0\big)\,\varphi_2\pi^2+\varphi_1\pi+\varphi_0\left(\frac{\varphi_2}{\psi_1}\pi+\left(\varphi_1-\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right)\frac{1}{\psi_1}\right.\\
\qquad\underline{\varphi_2\pi^2+\varphi_2\frac{\psi_1'}{\psi_1}\pi+\varphi_2\frac{\psi_0}{\psi_1}\pi+\varphi_2\frac{\psi_0'}{\psi_1}}\\
\qquad\left(\varphi_1-\varphi_2\frac{\psi_1'+\psi_0}{\psi_1}\right)\pi+\varphi_0-\varphi_2\frac{\psi_0'}{\psi_1}\\
\qquad\underline{\left(\varphi_1-\varphi_2\frac{\psi_1'+\psi_0}{\psi_1}\right)\pi+\left(\varphi_1-\varphi_2\frac{\psi_1'+\psi_0}{\psi_1}\right)\frac{\psi_0}{\psi_1}}\\
\qquad\qquad\varphi_0-\varphi_1\frac{\psi_0}{\psi_1}+\varphi_2\left\{\left(\frac{\psi_0}{\psi_1}\right)^2-\frac{\psi_0'\psi_1-\psi_0\psi_1'}{\psi_1^2}\right\}.
\end{array}
$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varphi_2\left\{\left(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\right)^2-\left(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\right)'\right\}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Before proceeding further, we may remark that the remainder, after internally dividing $\varphi_3(\varrho)\pi^3+\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$, can differ from that last above found only in respect of the remainder arising from the division of $\varphi_3(\varrho)\pi^3$ by the factor in question; hence we have now only to divide the term $\varphi_3(\varrho)\pi^3$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$, and add the remainder so found to (2.), in order to have the condition required for the third degree. Proceeding to the division, writing $\chi=\frac{\psi_0}{\psi_1}$, and omitting for the present φ_3, which, since the division is internal, can be replaced as an external factor in the remainder, we have

$$
\begin{array}{l}
\pi+\chi)\pi^3(\pi^2-\chi\pi+\chi^2-2\chi'\\
\qquad\underline{\pi^3+\chi\pi^2+2\chi'\pi+\chi''}\\
\qquad-\chi\pi^2-2\chi'\pi-\chi''\\
\qquad\underline{-\chi\pi^2-\chi^2\pi-\chi\chi'}\\
\qquad\quad(\chi^2-2\chi')\pi+\chi\chi'-\chi''\\
\qquad\quad\underline{(\chi^2-2\chi')\pi+\chi^3-2\chi\chi'}\\
\qquad\qquad\qquad-\chi^3+3\chi\chi'-\chi''.
\end{array}
$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_3(\varrho)\pi^3+\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varphi_2(\varrho)\left\{\left(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\right)^2-\left(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\right)'\right\}-\varphi_3(\varrho)\left\{\left(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\right)^3-3\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\left(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\right)'+\left(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\right)''\right\}=0, \quad (3.)$$

the identity of which with Mr. RUSSELL'S condition, given in p. 75 of his paper, I have verified.

For the fourth degree,

$$\begin{array}{l}
\pi+\chi)\pi^4(\pi^3-\chi\pi^2+(\chi^2-3\chi')\pi-(\chi^3-5\chi\chi'+3\chi'') \\
\quad\underline{\pi^4+\chi\pi^3+3\chi'\pi^2+3\chi''\pi+\chi'''} \\
\quad -\chi\pi^3-3\chi'\pi^2-3\chi''\pi-\chi''' \\
\quad \underline{-\chi\pi^3-\chi^2\pi^2-2\chi\chi'\pi-\chi\chi''} \\
\qquad (\chi^2-3\chi')\pi^2+(2\chi\chi'-3\chi'')\pi+(\chi\chi''-\chi''') \\
\qquad \underline{(\chi^2-3\chi')\pi^2+(\chi^3-3\chi\chi')\pi+(\chi^2\chi'-3\chi'^2)} \\
\qquad\quad -(\chi^3-5\chi\chi'+3\chi'')\pi-\chi^2\chi'+3\chi'^2+\chi\chi''-\chi''' \\
\qquad\quad \underline{-(\chi^3-5\chi\chi'+3\chi'')\pi-\chi^4+5\chi^2\chi'-3\chi\chi''} \\
\qquad\qquad \underline{\underline{\chi^4-6\chi^2\chi'-4\chi\chi''+3\chi\chi'^2-\chi'''}}
\end{array}$$

and

$$\chi^4-6\chi^2\chi'-4\chi\chi''+3\chi'^2-\chi'''=-\chi(-\chi^3+3\chi\chi'-\chi'')+(-\chi^3+3\chi\chi'-\chi'')';$$

or if R_1, R_2, .. be the remainders of $\pi(\pi+\chi)^{-1}$, $\pi^2(\pi+\chi)^{-1}$, .., we have

$$R_2=-\chi R_1+R_1',$$

$$R_3=-\chi R_2+R_2',$$

$$R_4=-\chi R_3+R_3'.$$

And generally, if

$$\pi^n\ (\pi+\chi)^{-1}=\ Q_n+\ R_n(\pi+\chi)^{-1},$$

then

$$\pi^{n+1}(\pi+\chi)^{-1}=\pi Q_n+\pi R_n(\pi+\chi)^{-1},$$

the remainder of which must be contained in the last term. Performing the actual division, and remembering that $\pi R_n=R_n\pi+R_n'$,

$$\begin{array}{l}
\pi+\chi)R_n\pi+R_n'(R_n \\
\qquad\quad \underline{R_n\pi+R_n\chi} \\
\qquad\qquad \underline{\underline{-\chi R_n+R_n'.}}
\end{array}$$

Hence we have generally,

$$R_{n+1}=-\chi R_n+R_n',$$

and consequently, remembering that $R_0=1$, we have the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\ldots\varphi_0(\varrho)$,

$$\varphi_0(\varrho)R_0+\varphi_1(\varrho)R_1+\ldots\varphi_n(\varrho)R_n=0, \quad \ldots\ldots\ldots\ldots \quad (4.)$$

where

$$R_{i+1}=-\frac{\psi_0(\varrho)}{\psi_1(\varrho)}R_i+\varrho\frac{d}{d\varrho}R_i=\left(-\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varrho\frac{d}{d\varrho}\right)R_i.$$

The law of the quotients is best seen by actual division. In case of $\varphi_2\pi^2+\varphi_1\pi+\varphi_0$, given above, the quotient may be written

$$\frac{1}{\psi_1}\times\begin{array}{|cc|c|}\hline \phi_1 & \phi_2 & \\ \hline 0 & 1 & \pi \\ 1 & -\frac{1}{\psi_1}(\psi_1'+\psi_0) & 1 \\ \hline \end{array} \quad \text{. . . . . (5.)}$$

For the case of a cubic function of π,

$$\psi_1\pi+\psi_0)\phi_3\pi^3+\phi_2\pi^2+\phi_1\pi+\phi_0\left(\phi_3\frac{1}{\psi_1}\pi^2+\left(\phi_2-\phi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\frac{1}{\psi_1}\pi+\left(\phi_1-\phi_2\frac{\psi_1'+\psi_0}{\psi_1}+\phi_3\frac{2\psi_1'^2+3\psi_0\psi_1'+\psi_0^2-\psi_1\psi''-2\psi_1\psi_0'}{\psi_1^2}\right)\right.$$

$$\frac{\phi_3\pi^3+2\phi_3\frac{\psi_1'}{\psi_1}\pi^2+\phi_3\frac{\psi_1''}{\psi_1}\pi+\phi_3\frac{\psi_0}{\psi_1}\pi^2+2\phi_3\frac{\psi_0'}{\psi_1}\pi+\phi_3\frac{\psi_0''}{\psi_1}}{\left(\phi_2-\phi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\pi^2+\left(\phi_1-\phi_3\frac{\psi_1''+2\psi_0'}{\psi_1}\right)\pi+\left(\phi_0-\phi_3\frac{\psi_0''}{\psi_1}\right)}$$

$$\frac{\left(\phi_2-\phi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\pi^2+\left(\phi_2-\phi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\frac{\psi_1'}{\psi_1}\pi+\left(\phi_2-\phi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\frac{\psi_0}{\psi_1}\pi+\left(\phi_2-\phi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\frac{\psi_0'}{\psi_1}}{\left(\phi_1-\phi_2\frac{\psi_1'+\ {}_0}{\psi_1}+\phi_3\frac{2\psi_1'^2+3\psi_0\psi_1'+\psi_0^2-\psi_1\psi_1''-2\psi_1\psi_0'}{\psi_1^2}\right)\pi+\left(\phi_0-\phi_2\frac{\psi_0'}{\psi_1}+\phi_3\frac{2\psi_1'\psi_0'+\psi_0\psi_0'-\psi_1\psi_0''}{\psi_1^2}\right).}$$

The quotient of which may be written

$$\frac{1}{\psi_1}\times\begin{array}{|ccc|c|}\hline \phi_1 & \phi_2 & \phi_3 & \\ \hline 0 & 0 & 1 & \pi^2 \\ 0 & 1 & -\frac{1}{\psi_1}(2\psi_1'+\psi_0) & \pi \\ 1 & -\frac{1}{\psi_1}(\psi_1'+\psi_0) & \frac{1}{\psi_1^2}\begin{vmatrix} 2\psi_1'+\psi_0 & \psi_1 \\ \psi_1''+2\psi_0' & 2\psi_1'+\psi_0 \end{vmatrix} & 1 \\ \hline \end{array} \quad \text{(6.)}$$

Similarly, if the division be performed in the case of the quartic function, we shall find for the quotient of $(\phi_4\pi^4+\phi_3\pi^3+\phi_2\pi^2+\phi_1\pi+\phi_0)(\psi_1\pi+\psi_0)^{-1}$,

$$\frac{1}{\psi_1}\times\begin{array}{|cccc|c|}\hline \phi_1 & \phi_2 & \phi_3 & \phi_4 & \\ \hline 0 & 0 & 0 & 1 & \pi^3 \\ 0 & 0 & 1 & -\frac{1}{\psi_1}(3\psi_1'+\psi_0) & \pi^2 \\ 0 & 1 & -\frac{1}{\psi_1}(2\psi_1'+\psi_0) & \frac{1}{\psi_1^2}\begin{vmatrix} 3\psi_1'+\psi_0 & \psi_1 \\ 3\psi_1''+3\psi_0' & 2\psi_1'+\psi_0 \end{vmatrix} & \pi \\ 1 & -\frac{1}{\psi_1}(\psi_1'+\psi_0) & \frac{1}{\psi_1^2}\begin{vmatrix} 2\psi_1'+\psi_0 & \psi_1 \\ \psi_1''+2\psi_0' & \psi_1'+\psi_0 \end{vmatrix} & -\frac{1}{\psi_1^3}\begin{vmatrix} 3\psi_1'+\psi_0 & \psi_1 & 0 \\ 3\psi_1''+3\psi_0' & 2\psi_1'+\psi_0 & \psi_1 \\ \psi_1'''+3\psi_0'' & \psi_1''+\psi_0' & \psi_1'+\psi_0 \end{vmatrix} & 1 \\ \hline \end{array} \quad \text{(7.)}$$

And likewise in general the quotient of $(\varphi_n\pi^n+\varphi_{n-1}\pi^{n-1}+\,.\,.\,\varphi_0)(\psi_1\pi+\psi_0)^{-1}$ will be represented by a square table giving for the coefficient of φ_n

$$(-)^{n-1}\frac{1}{\psi_1^{n-1}}\times\left|\begin{array}{lllll} \frac{i-1}{1} & \psi_1'+\psi_0 & \psi_1 & .\,. & 0 \\ \frac{(i-1)(i-2)}{1.2} & \psi_1''+\frac{i-1}{1}\psi_0' & \frac{i-2}{1}\psi_1'+\psi_0 & .\,. & 0 \\ & \vdots & \vdots & \vdots & \vdots \\ & \psi_1^{(i-1)}+\frac{i-1}{1}\psi_0^{(i-2)} & \psi_1^{(i-2)}+\frac{i-2}{1}\psi_0^{(i-2)} & .\,. & \psi_1'+\psi_0 \end{array}\right| \qquad (8.)$$

§ 2. *External division of the π-arrangement by a linear factor.*

I next investigate the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an *external* factor of

$$\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\,.\,.\,\varphi_0(\varrho).$$

Performing the actual divisions, we have in the case of $n=1$,

$$\begin{array}{l} \psi_1\pi+\psi_0\big)\,\varphi_1\pi+\varphi_0\,\Big(\frac{\varphi_1}{\psi_1} \\ \qquad\qquad \underline{\varphi_1\pi+\psi_1\left(\frac{\varphi_1}{\psi_1}\right)'+\psi_0\frac{\varphi_1}{\psi_1}} \\ \qquad\qquad \underline{\varphi_0-\psi_1\left(\frac{\varphi_1}{\psi_1}\right)'-\psi_0\frac{\varphi_1}{\psi_1};} \end{array}$$

or, as the remainder may be more conveniently written,

$$\varphi_0-\varphi_1'+\varphi_1\frac{\psi_1'-\psi_0}{\psi_1}.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\qquad (1.)$$

Again, in the case of $n=2$,

$$\begin{array}{l} \psi_1\pi+\psi_0\big)\,\varphi_2\pi^2+\varphi_1\pi+\varphi_0\,\Big(\frac{\varphi_2}{\psi_1}\pi+\frac{1}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\} \\ \qquad\qquad \underline{\varphi_2\pi^2+\psi_1\left(\frac{\varphi_2}{\psi_1}\right)'\pi+\psi_0\frac{\varphi_2}{\psi_1}\pi} \\ \qquad \left\{\varphi_1-\psi_1\left(\frac{\varphi_2}{\psi_1}\right)'-\psi_0\frac{\varphi_2}{\psi_1}\right\}\pi+\varphi_0; \end{array}$$

or, transforming the remainder as in the former case, and continuing the division,

$$\begin{array}{l} \left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\pi+\varphi_0 \\ \underline{\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\pi+\psi_1\left[\frac{1}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\right]'+\frac{\psi_0}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}} \\ \qquad\qquad \varphi_0-\psi_1\left[\frac{1}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\right]'-\frac{\psi_0}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}, \end{array}$$

which also may be transformed as follows:—

$$\phi_0-\phi_1'+\phi_2''+(\phi_1-2\phi_2')\frac{\psi_1'-\psi_0}{\psi_1}+\phi_2\left\{\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)^2-\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)'\right\}. \quad . \quad . \quad . \quad (2.)$$

A similar process of division will be found, in the case of $n=3$, to lead to the following remainder:—

$$\left.\begin{aligned}&\phi_0-\phi_1'+\phi_2''-\phi_3'''+(\phi_1-2\phi_2'+3\phi_3'')\frac{\psi_1'-\psi_0}{\psi_1}+(\phi_2-3\phi_3')\left\{\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)^2-\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)'\right\}\\&\quad+\phi_3\left\{\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)^3-3\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)'+\left(\frac{\psi_1'-\psi_0}{\psi_1}\right)''\right\}.\end{aligned}\right\} \quad (3.)$$

If Ψ_1, Ψ_2, .. represent the ψ-functions, coefficients of the ϕs in this expression, the law of their formation will be found to be as follows:—

$$\begin{aligned}\Psi_1 &=\frac{\psi_1'-\psi_0}{\psi_1},\\ \Psi_2 &=\Psi_1^2-\Psi_1',\\ \Psi_3 &=\Psi_1\Psi_2-\Psi_2',\\ &\cdots\cdots\end{aligned}$$

And generally we may write

$$\Psi_{i+1}=\Psi_1\Psi_i-\Psi_i'.$$

And if R_1, R_2, .. represent the remainders in the cases of $n=1$, $n=2$, .. respectively, we have

$$\begin{aligned}R_1&=\phi_0-\phi_1'+\phi_1\Psi_1,\\ R_2&=\phi_0-\phi_1'+\phi_2''+(\phi_1-2\phi_2')\Psi_1+\phi_2\Psi_2,\\ R_3&=\phi_0-\phi_1'+\phi_2''-\phi_3'''+(\phi_1-2\phi_2'+3\phi_3'')\Psi_1+(\phi_2-3\phi_3')\Psi_2+\phi_3\Psi_3;\end{aligned}$$

whence

$$\begin{aligned}R_2&=R_1+\phi_2''-2\phi_2'\Psi_1+\phi_2\Psi_2,\\ R_3&=R_2-\phi_3'''+3\phi_3''\Psi_1-3\phi_3'\Psi_2+\phi_3\Psi_3.\end{aligned}$$

With a view to forming the expression for R_n, let the symbol $\binom{1}{0}$ affixed to R_i signify that in the expression for R_n the suffixes of the ϕs have all been increased by unity. Then, by the principle of division,

$$\begin{aligned}R_{n+} &=\phi_0-\psi_1\left[\frac{1}{\psi_1}R_n\binom{1}{0}\right]'-\frac{\psi_0}{\psi_1}R_n\binom{1}{0}\\ &=\phi_0-R_n'\binom{1}{0}+\Psi_1R_n\binom{1}{0}\\ &=\phi_0-\left(R_{n-1}-\phi_n^{(n)}+\frac{n}{1}\phi_n^{(n-1)}\Psi_1-\frac{n(n-1)}{1.2}\phi_n^{(n-2)}\Psi_2+..\right)'\binom{1}{0}\\ &\qquad+\Psi_1\left(R_{n-1}-\phi_n^{(n)}+\frac{n}{1}\phi_n^{(n-1)}\Psi_1-\frac{n(n-1)}{1.2}\phi_n^{(n-2)}\Psi_2+..\right)\binom{1}{0}\\ &=\phi_0-\left(R_{n-1}'-\Psi_1R_{n-1}\right)\binom{1}{0}+\phi_{n+1}^{(n+1)}-\frac{n+1}{1}\phi_{n+1}^{(n)}+..;\end{aligned}$$

and in this expression

$$\phi_0-\left(R'_{n-1}-\Psi_1 R_{n-1}\right)\binom{1}{0}=R_n,$$

while the general term of the ϕ-series, *i. e.* the coefficient of $\phi^{(n-r)}_{n+1}$, will be

$$(-)^{r-1}\left\{\frac{n(n-1)..(n-r+1)}{1.2..r}\Psi'_r-\frac{n(n-1)..(n-r)}{1.2..(r+1)}\Psi_{r+1}-\frac{n(n-1)..(n-r+1)}{1.2..r}\Psi_1\Psi_r\right\}$$

$$=(-)^{r-1}\frac{n(n-1)..(n-r+1)}{1.2..r}\left\{\Psi'_r-\Psi_1\Psi_r-\frac{n-r}{r+1}\Psi_{r+1}\right\}$$

$$=(-)^r\frac{(n+1)n(n-1)..(n-r+1)}{1.2..(r+1)}\Psi_{r+1},$$

which proves the general case; so that generally

$$R_{n+1}=R_n\pm\phi^{(n+1)}_{n+1}\mp\frac{n+1}{1}\phi^{(n)}_{n+1}\Psi_1\pm\frac{(n+1)n}{1.2}\phi^{(n)}_{n+1}\Psi_2\mp..,\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

the upper or lower sign being taken according as $(n+1)$ is even or odd; where R_{n+1} is the remainder after external division $\phi_{n+1}(\varrho)\pi^{n+1}+\phi_n(\varrho)\pi^n+..\phi_0$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$.

For the quotients Q_1, Q_2, .. we have immediately

$$Q_1=\frac{1}{\psi_1}\phi_1,$$

$$Q_2=\frac{1}{\psi_1}\left\{\phi_2\pi+R_1\binom{1}{0}\right\},$$

$$Q_3=\frac{1}{\psi_1}\left\{\phi_3\pi^2+R_1\binom{2}{0}\pi+R_2\binom{1}{0}\right\}.$$

$$.. \quad ...$$

$$Q_n=\frac{1}{\psi_1}\left\{\phi_n\pi^{n-1}+R_1\binom{n-1}{0}\pi^{n-2}+R_2\binom{n-2}{0}\pi^{n-3}+..R_{n-1}\binom{1}{0}\right\}.$$

This completes the solution of the problem of division by a linear factor, both internal and external.

§ 3. *To divide $\Sigma_{n=0}^{n=N}\phi_n\pi^n$ internally by $\Sigma_{m=0}^{m=M}\psi_m\pi^m$.*

The first term in the quotient will obviously be

$$\frac{\phi_N}{\psi_M}\pi^{N-M},\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

and the product of this into the divisor may, by means of LEIBNITZ's theorem, be written thus:

$$\frac{\phi_N}{\psi_m}\Sigma_{p=0}^{p=N-M}[N-M,p]\ \Sigma_{m=0}^{m=M}\psi_m^{(N-M-p)}\pi^{m+p},\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

where $\psi_m^{(N-M-p)}$ means the result of the operation π^{N-M-p} or ψ_m alone, and

$$[N-M,p]=\frac{(N-M)(N-M-1)..(N-M-p+1)}{1.2..p}.$$

Then the remainder after subtraction from the dividend may be written thus:

$$\Sigma_{m+p=0}^{m+p=N-1}\left\{\phi_{m+p}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\phi_N[N-M,p]\psi_m^{(N-M-p)}\right\}\pi^{m+p}, \quad . \quad . \quad . \quad (3.)$$

since the coefficient of π^N vanishes. With a view to the second term in the dividend, the first term of the remainder (3.) is

$$\left\{\phi_{N-1}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\phi_N[N-M,p]\psi_m^{(N-M-p)}\right\}\pi^{N-1}, \quad . \quad . \quad . \quad (4.)$$

in which the limits of p and m are subject to the further condition $p+m=N-1$. The terms under the sign of summation will be evaluated hereafter. Putting the expression

$$(4)=\Phi_1\pi^{N-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

and, for the sake of symmetry,

$$\phi_N=\Phi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

the first and second terms in the quotient will be $\frac{\Phi_0}{\psi_M}\pi^{N-M}$, and $\frac{\Phi_1}{\psi_M}\pi^{N-M-1}$, respectively; and, in the same manner as (2.), the product of the second term of the quotient into the divisor may be written thus,

$$\frac{\Phi_1}{\psi_M}\Sigma_{p_1=0}^{p_1=N-M-1}\Sigma_{m=0}^{m=M}[N-M-1,p]\psi_m^{(N-M-p-1)}\pi^{m+p_1}, \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

and the remainder thus:

$$\left.\begin{array}{l}\Sigma_{m+p=0}^{m+p=N-2}\left\{\phi_{m+p}-\dfrac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Phi_0[N-M,p]\psi_m^{(N-M-p)}\right\}\pi^{m+p}\\ -\Sigma_{m+p_1=0}^{m+p_1=N-2}\left\{\dfrac{1}{\psi_M}\Sigma_{p_1=0}^{p_1=N-M-1}\Sigma_{m=0}^{m=M}\Phi_1[N-M--1,p_1]\psi_m^{(N-M-p_1-1)}\right\}\pi^{m+p_1}.\end{array}\right\} \quad . \quad (8.)$$

But since, when $p_1=N-M$, $[N-M-1,p_1]=0$, we may, without altering the value of (8.), change the superior limit of p_1 from $N-M-1$, to $N-M$; and by this means we may write the remainder (8.) in the following form:

$$\Sigma_{m+p=0}^{m+p=N-2}\left\{\phi_{m+p}-\frac{1}{\psi_m}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}(\Phi_0[N-M,p]\psi_m^{(N-M-p)}+\Phi_1[N-M-1,p]\psi_m^{(N-M-p-1)})\right\}\pi^{m+p}. \quad (9.)$$

Similarly, calling the first term of (9.) $\Phi_2\pi^{N-2}$, the third term in the quotient will be $\frac{\Phi_2}{\psi_M}\pi^{N-M-2}$, and the corresponding remainder

$$\left.\begin{array}{l}\Sigma_{m+p=0}^{m+p=N-3}\left\{\phi_{m+p}-\dfrac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}(\Phi_0[N-M,p]\psi_m^{(N-M-p)}+\Phi_1[N-M-1,p]\psi_m^{(N-M-p-1)}\right.\\ \qquad\left.+\Phi_2[N-M-2,p]\psi_m^{(N-M-p-2)})\right\}\pi^{m+p};\end{array}\right\}(10.)$$

and so generally the $(r+1)$th term in the quotient will be $\frac{\Phi_r}{\psi_M}\pi^{N-M-r}$, where

$$\Phi_r=\phi_{N-r}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Sigma_{q=0}^{q=r-1}\Phi_q[N-M-q,p]\psi_m^{(M-N-p-q)}, \quad . \quad . \quad . \quad . \quad (11.)$$

and the $(r+1)$th remainder

$$\Sigma_{m+p=0}^{m+p=\mathrm{N}-r-1}\left\{\phi_{m+p}-\Sigma_{p=0}^{p=\mathrm{N}-\mathrm{M}}\Sigma_{m=0}^{m=\mathrm{M}}\Sigma_{q=0}^{q=r}\Phi_q[\mathrm{N}-\mathrm{M}-q,p]\psi_m^{(\mathrm{N}-\mathrm{M}-p-q)}\right\}\pi^{m+p}. \quad . \quad . \quad (12.)$$

The final remainder is the $(\mathrm{N}-\mathrm{M}+1)$th; and the expression will be derived from (10.) by replacing r by $\mathrm{N}-\mathrm{M}$.

It remains to develope the terms under the sign of summation in the expressions for the Φs. In the first place $\Phi_0=\phi_\mathrm{N}$ simply. In the case of Φ_1, the limiting values of p and m are

$$\begin{aligned} p&=0,\ 1,\ ..\ \mathrm{N}-\mathrm{M},\\ m&=0,\ 1,\ ..\ \mathrm{M},\\ p+m&=\mathrm{N}-1. \end{aligned}$$

These give as the only admissible values

$$\begin{array}{lll} p=\mathrm{N}-\mathrm{M} & , & \mathrm{N}-\mathrm{M}-1,\\ m= & \mathrm{M}-1, & \mathrm{M} \quad , \end{array}$$

and consequently

$$\Phi_1=\phi_{\mathrm{N}-1}-\frac{\Phi_0}{\psi_\mathrm{M}}\{\psi_{\mathrm{M}-1}+(\mathrm{N}-\mathrm{M})\psi'_\mathrm{M}\}.$$

In the case of Φ_2, the only admissible values are

$$\begin{array}{llll} p=\mathrm{N}-\mathrm{M} & , \ \mathrm{N}-\mathrm{M}-1, & \mathrm{N}-\mathrm{M}-2,\\ m= \quad \mathrm{M}-2, & \mathrm{M}-1, & \mathrm{M} \quad , \end{array}$$

giving

$$\begin{aligned}\Phi_2=\phi_{\mathrm{N}-2}-\frac{1}{\psi_\mathrm{M}}\Big\{&\Phi_0\left(\psi_{\mathrm{M}-2}+(\mathrm{N}-\mathrm{M})\psi'_{\mathrm{M}-1}+\frac{(\mathrm{N}-\mathrm{M})(\mathrm{N}-\mathrm{M}-1)}{1.2}\psi''_\mathrm{M}\right)\\ &+\Phi_1\left(\psi_{\mathrm{M}-1}+(\mathrm{N}-\mathrm{M}-1)\psi'_\mathrm{M}\right)\Big\}.\end{aligned}$$

Before proceeding further, it may be well to illustrate these formulæ by an example.

Taking the case of $\mathrm{N}=4$, $\mathrm{M}=3$, we may determine the quotient and last remainder of internal division of

$$\phi_4(\varrho)\pi^4+\phi_3(\varrho)\pi^3+\phi_2(\varrho)\pi^2+\phi_1(\varrho)\pi+\phi_0(\varrho)$$

by

$$\psi_3(\varrho)\pi^3+\psi_2(\varrho)\pi^2+\psi_1(\varrho)\pi+\psi_0(\varrho),$$

and thence the conditions that the latter may be an internal factor of the former.

By the formulæ given above, we have

$$\begin{aligned}\Phi_0&=\phi_4,\\ \Phi_1&=\phi_3-\frac{\phi_4}{\psi_2}\{\psi_1+2\psi'_2\},\\ \Phi_2&=\phi_2-\frac{1}{\psi_2}\left\{\phi_4(\psi_0+2\psi'_1+\psi''_2)+\left(\phi_3-\frac{\phi_4}{\psi_2}(\psi_1+2\psi'_2)\right)(\psi_1+\psi'_2)\right\},\end{aligned}$$

which will determine the quotient

$$\frac{1}{\psi_2}(\Phi_0\pi^2+\Phi_1\pi+\Phi_2).$$

The last remainder is

$$\Sigma_{m+p=0}^{m+p=1}\left\{\phi_{m+p}-\frac{1}{\psi_2}\Sigma_{p=0}^{p=2}\Sigma_{m=0}^{m=2}\Sigma_{q=0}^{q=2}\Phi_q[\mathrm{N}-\mathrm{M}-q,p]\psi_m^{(\mathrm{N}-\mathrm{M}-p-q)}\right\}\pi^{m+p};$$

whence for

$$m+p=1,\ i.\ e.\ p=0,\ m=1,\ \text{or}\ p=1,\ m=0,$$

we have

$$\begin{array}{llll} & q=0 & q=1 & q=2 \\ p=0, & [2,0]=1, & [1,0]=1, & [0,0]=1, \\ p=1, & [2,1]=2, & [1,1]=1, & [0,1]=0. \end{array}$$

Hence for $m=1$ the above expression gives

$$\Phi_0\psi_1''+\Phi_1\psi_1'+\Phi_2\psi_1,$$

and for $m=0$ it gives

$$2\Phi_0\psi_0'+\Phi_1\psi_0.$$

Again, for $m+p=0$, $i.\ e.\ p=0$, $m=0$, we have

$$\Phi_0\psi_0''+\Phi_1\psi_0'+\Phi_2\psi_0.$$

Hence the total remainder is

$$\left\{\phi_1-\frac{1}{\psi_2}(\Phi_0\psi_1''+\Phi_1\psi_1'+\Phi_2\psi_1+2\Phi_0\psi_0'+\Phi_1\psi_0)\right\}\pi+\left\{\phi_0-\frac{1}{\psi_2}(\Phi_0\psi_0''+\Phi_1\psi_0'+\Phi_2\psi_0)\right\}.$$

It may be useful to compare these results with the actual division, in the above example.

$$\psi_2\pi^2+\psi_1\pi+\psi_0\,\big)\,\phi_4\pi^4+\phi_3\pi^3+\phi_2\pi^2+\phi_1\pi+\phi_0\left(\frac{\Phi_0}{\psi_2}\pi^2+\frac{\Phi_1}{\psi_2}\pi+\frac{\Phi_2}{\psi_2}\right.$$

$$\left.\begin{array}{l} \phi_4\pi^4+2\phi_4\frac{\psi_2'}{\psi_2}\pi^3+\phi_4\frac{\psi_2''}{\psi_2}\pi^2 \\ \quad+\ \phi_4\frac{\psi_1}{\psi_2}\pi^3+2\phi_4\frac{\psi_1'}{\psi_2}\pi^2+\phi_4\frac{\psi_1''}{\psi_2}\pi \\ \qquad\qquad+\phi_4\frac{\psi_0}{\psi_2}\pi^2+2\phi_4\frac{\psi_0'}{\psi_2}\pi+\phi_4\frac{\psi_0''}{\psi_2} \end{array}\right\}$$

$$\Phi_1\pi^3+\left(\phi_2-\frac{\phi_4}{\psi_2}(\psi_2''+2\psi_1'+\psi_0)\right)\pi^2+\left(\phi_1-\frac{\phi_4}{\psi_2}(\psi_1''+2\psi_0')\right)\pi+\left(\phi_0-\frac{\phi_4}{\psi_2}\psi_0''\right)$$

$$\left.\begin{array}{l} \Phi_1\pi^3+\Phi_1\frac{\psi_2'}{\psi_2}\pi^2 \\ \qquad+\Phi_1\frac{\psi_1}{\psi_2}\pi^2+\Phi_1\frac{\psi_1'}{\psi_2}\pi \\ \qquad\qquad+\Phi_1\frac{\psi_0}{\psi_2}\pi+\Phi_1\frac{\psi_0'}{\psi_2} \end{array}\right\}$$

$$\Phi_2\pi^2+\left\{\phi_1-\frac{1}{\psi_2}\Phi_0(\psi_1''+2\psi_0')-\frac{1}{\psi_2}\Phi_1(\psi_1'+\psi_0)\right\}\pi+\left\{\phi_0-\frac{1}{\psi_2}\Phi_0\psi_0''-\frac{1}{\psi_2}\Phi_1\psi_0'\right\}$$

$$\Phi_2\pi^2+\frac{1}{\psi_2}\Phi_2\psi_1\pi\qquad\qquad+\frac{1}{\psi_2}\Phi_2\psi_0$$

$$\left\{\phi_1-\frac{1}{\psi_2}\Phi_0(\psi_1''+2\psi_0')-\frac{1}{\psi_2}\Phi_1(\psi_1'+\psi_0)-\frac{1}{\psi_2}\Phi_2\psi_1\right\}\pi+\left\{\phi_0-\frac{1}{\psi_2}\Phi_0\psi_0''-\frac{1}{\psi_2}\Phi_1\psi_0-\frac{1}{\psi_2}\Phi_2\psi_0\right\},$$

which agrees with the result before found.

Returning to the Φ functions, and writing for convenience the symbolical expression

$$\Sigma_{p=0}^{p=N-M-r}\Sigma_{m=0}^{m=M}[N-M-r,\,p]\psi_m^{(N-M-p-r)}=S_s[N-M-r,\,p]\psi_m^{(N-M-p-r)},$$
$$p+m=N-s,$$

where the suffix s indicates the number of units whereby the sum $p+m$ is less than N, we have

$$\Phi_0=\phi_N,$$

$$\psi_M\Phi_1=\psi_M\phi_{N-1}-S_1\Phi_0[N-M,\,p]\psi_m^{(N-M-p)}$$
$$=\begin{vmatrix} \phi_{N-1}S_1[N-M,\,p]\psi_m^{(N-M-p)} & \\ \phi_N & \psi_M \end{vmatrix}, \quad \ldots\ldots\ldots \quad (13.)$$

$$\psi_M^2\Phi_2=\psi_M^2\phi_{N-2}-S_2(\Phi_0[N-M,\,p]\psi_m^{(N-M-p)}+\Phi_1\;[N-M-1,\,p]\psi_m^{(N-M-p-1)})$$
$$=\psi_M^2\phi_{N-2}-\Phi_0S_2\,[N-M,\,p]\psi_m^{(N-M-p)}-\Phi_1S_2[N-M-1,\,p]\psi_m^{(N-M-p-1)}$$
$$=\begin{vmatrix} \phi_{N-2} & S_2[N-M-1,\,p]\psi_m^{(N-M-p-1)} & S_2[N-M,\,p]\psi_m^{(N-M-p)} \\ \phi_{N-1} & S_1[N-M-1,\,p]\psi_m^{(N-M-p-1)} & S_1[N-M,\,p]\psi_m^{(N-M-p)} \\ \phi_N & 0 & S_0[N-M,\,p]\psi_m^{(N-M-p)} \end{vmatrix}; \quad \ldots \quad (14.)$$

and generally,

$$\psi_M^r\Phi_r=\begin{vmatrix} \phi_{N-r} & S_r\;[N-M-r+1,\,p]\psi_m^{(N-M-p-r+1)} & \ldots & S_r\;[N-M-1,\,p]\psi_m^{(N-M-p-1)} & S_r[N-M,\,p]\psi_m^{(N-M-p)} \\ \phi_{N-r+1} & S_{r-1}[N-M-r+1,\,p]\psi_m^{(N-M-p-r+1)} & \ldots & S_{r-1}[N-M-1,\,p]\psi_m^{(N-M-p-1)} & S_r[N-M,\,p]\psi_m^{(N-M-p)} \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \phi_N & 0 & \ldots & 0 & S_0[N-M,\,p]\psi_m^{(N-M-p)} \end{vmatrix}. \quad (15.)$$

These formulæ give, for the example discussed above,

$$\Phi_0=\phi_4$$

$$\psi_2\Phi_1=\begin{vmatrix} \phi_3 & \psi_1+2\psi_2' \\ \phi_4 & \psi_2 \end{vmatrix}$$

$$\psi_2^2\Phi_2=\begin{vmatrix} \phi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & \psi_2 & \psi_1+2\psi_2' \\ \phi_4 & 0 & \psi_2 \end{vmatrix}.$$

And for the final remainder, the coefficient of π,

$$\psi_2^3\left\{\phi_1-\frac{1}{\psi_2}\Phi_0(\psi_1''+2\psi_0')-\frac{1}{\psi_2}\Phi_1(\psi_1'+\psi_0)-\frac{1}{\psi_2}\Phi_2\psi_1\right\}=\begin{vmatrix} \phi_1 & \psi_1 & \psi_0+\psi_1' & 2\psi_0'+\psi_1'' \\ \phi_2 & \psi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & 0 & \psi_2 & \psi_1+2\psi_2' \\ \phi_4 & 0 & 0 & \psi_2 \end{vmatrix},$$

and the terms independent of $\pi=$

$$\psi_2^3\left\{\phi_0-\frac{1}{\psi_2}(\Phi_0\psi_3''+\Phi_1\psi_0'+\Phi_2\psi_0)\right\}=\begin{vmatrix}\phi_0 & \psi_0 & \psi_0' & \psi_0'' \\ \phi_2 & \psi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & 0 & \psi_1 & \psi_1+2\psi_2' \\ \phi_4 & 0 & 0 & \psi_2\end{vmatrix},$$

both of which may be comprised under the single formula

$$\begin{vmatrix}\pi & \phi_0 & \psi_0 & \psi_0' & \psi_0'' \\ 1 & \phi_1 & \psi_1 & \psi_0+\psi_1' & 2\psi_0'+\psi_1'' \\ 0 & \phi_2 & \psi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ 0 & \phi_3 & 0 & \psi_2 & \psi_1+2\psi_2' \\ 0 & \phi_4 & 0 & 0 & \psi_2\end{vmatrix}$$

§ 4. *To divide* $\Sigma_{n=0}^{n=N}\phi_n\pi^n$ *externally by* $\Sigma_{m=0}^{m=M}\psi_m\pi^m$.

The first term in the quotient will be

$$\frac{\phi_N}{\psi_M}\pi^{N-M}, \quad \ldots \ldots \quad (1.)$$

the product of this into the divisor

$$\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\phi_N}{\psi_M}\right)^{(m-p)}\pi^{N-M+p}, \quad \ldots \ldots \quad (2.)$$

and the remainder

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\phi_N}{\psi_M}\right)^{(m-p)}\pi^{N-M+p}; \quad \ldots \ldots \quad (3.)$$

the first term of which is

$$\left.\begin{aligned}&\left\{\phi_{N-1}-\Sigma_{m=0}^{m=M}\psi_m[m,M-1]\left(\frac{\phi_N}{\psi_M}\right)^{(m-M+1)}\right\}\pi^{N-1}\\ =&\left\{\phi_{N-1}-\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)-M\psi_M\left(\frac{\phi_N}{\psi_M}\right)'\right\}\pi^{N-1}.\end{aligned}\right\} \quad \ldots \ldots \quad (4.)$$

Let $\phi_N=\Phi_0$, and let the coefficient of π^{N-1} above written $=\Phi_1$; then the next term in the quotient will be $\frac{\Phi_1}{\psi_M}\pi^{N-M-1}$, and the product of this into the divisor will take the same form as (2.), writing only Φ_1 for Φ_0, or ϕ_N. The remainder will be the same as (3.), with the addition of the term

$$-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\Phi_1}{\psi_M}\right)^{m-p}\pi^{N-M+p-1}, \quad \ldots \ldots \quad (5.)$$

and the first term of the entire remainder will be

$$\left.\begin{aligned}&\left\{\phi_{N-2}-\Sigma_{m=0}^{m=M}\psi_m\left[[m,M-2]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+2)}+[m,M-1]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+1)}\right]\right\}\pi^{N-2}\\ =&\left\{\phi_{N-2}-\psi_{M-2}\left(\frac{\Phi_0}{\psi_M}\right)-(M-1)\psi_{M-1}\left(\frac{\Phi_0}{\psi_M}\right)'-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)''\right.\\ &\left.\qquad-\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)-M\psi_M\left(\frac{\Phi_1}{\psi_M}\right)'\right\}\pi^{N-2};\end{aligned}\right\} \quad \ldots \quad (6.)$$

and if we make this expression $=\Phi_2\pi^{N-2}$, the next remainder will be

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left\{\left(\frac{\Phi_0}{\psi_m}\right)^{(m-p)}\pi^2+\left(\frac{\Phi_1}{\psi_m}\right)^{(m-p)}\pi+\left(\frac{\Phi_0}{\psi_m}\right)^{(m-p)}\right\}\pi^{N-M+p-3}, \quad (7.)$$

the first term of which is

$$\left.\begin{aligned}&\left\{\phi_{N-3}-\Sigma_{m=0}^{m=M}\psi_m\left[[m,M-3]\left(\frac{\Phi_0}{\psi_M}\right)^{m-M+3}+[m,M-2]\left(\frac{\Phi_1}{\psi_M}\right)^{m-M+2}+[m,M-1]\left(\frac{\Phi_0}{\psi_M}\right)^{m-M+1}\right]\right\}\pi^{N-2}\\ &=\left\{\phi_{N-3}-\psi_{M-3}\left(\frac{\Phi_0}{\psi_m}\right)-(M-2)\psi_{M-2}\left(\frac{\Phi_0}{\psi_m}\right)'-\frac{(M-1)(M-2)}{1.2}\psi_{M-1}\left(\frac{\Phi_0}{\psi_m}\right)''-\frac{M(M-)(M-2)}{1.2.3}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)'''\right.\\ &\qquad-\psi_{M-2}\left(\frac{\Phi_1}{\psi_M}\right)-(M-1)\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)'-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_1}{\psi_M}\right)''\\ &\qquad\left.-\psi_{M-1}\left(\frac{\Phi_2}{\psi_M}\right)-M\psi_M\left(\frac{\Phi_2}{\psi_M}\right)'\right\}\pi^{N-3};\end{aligned}\right\}(8.)$$

and generally the $(r+1)$th term in the quotient will be

$$\frac{\Phi_r}{\psi_M}\pi^{N-M-r},$$

and the $(r+1)$th remainder

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\Sigma_{q=0}^{q=r}\left(\frac{\Phi_q}{\psi_M}\right)^{(m-p)}\pi^{N-M+p-q}. \quad . \quad . \quad . \quad . \quad (9.)$$

The formation of the Φs is as follows:—

$$\left.\begin{aligned}&\Phi_0=\phi_N,\\ &\Phi_1=\phi_{N-1}-\Sigma_{m=0}^{m=M}\psi_m\,[m,M-1]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+1)},\\ &\Phi_2=\phi_{N-2}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+2)}\right\},\\ &\Phi_3=\phi_{N-3}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_2}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+2)}\right.\\ &\qquad\left.+[m,M-3]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+3)}\right\},\\ &\quad\cdots\cdots\cdots\cdots\cdots\\ &\Phi_s=\phi_{N-s}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_{s-1}}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_{s-2}}{\psi_M}\right)^{(m-M+2)}\right.\\ &\qquad\left.+-[m,1]\left(\frac{\Phi_{s-M+1}}{\psi_M}\right)^{(M-1)}+[m,0]\left(\frac{\Phi_{s-M}}{\psi_M}\right)^{(M)}\right\}.\end{aligned}\right\} \quad . \quad (10.)$$

The final remainder is given by the formula

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\Sigma_{q=0}^{q=N-M}\left(\frac{\Phi_q}{\psi_M}\right)^{(m-p)}\pi^{N-M+p-q}; \quad . \quad . \quad . \quad (11.)$$

and the general term of this π^{N-s} is to be found as follows:

$$N-M+p-q=N-s,$$

i. e.

$$p-q=M-s.$$

Then we have for

$$
\left.
\begin{array}{l}
p=\mathrm{M}, \qquad q=s, \\
\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m, \mathrm{M}\ ]\left(\dfrac{\Phi_s}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M})}=\Phi_s; \\
p=\mathrm{M}-1,\ q=s-1, \\
\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m, \mathrm{M}-1]\left(\dfrac{\Phi_{s-1}}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M}+1)}=\mathrm{M}\psi_\mathrm{M}\left(\dfrac{\Phi_{s-1}}{\psi_\mathrm{M}}\right)'+\psi_{\mathrm{M}-1}\left(\dfrac{\Phi_{s-1}}{\psi_\mathrm{M}}\right); \\
p=\mathrm{M}-2,\ q=s-2, \\
\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m, \mathrm{M}-2]\left(\dfrac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M}+2)}=\dfrac{\mathrm{M}(\mathrm{M}-1)}{1.2}\psi_\mathrm{M}\left(\dfrac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)''+(\mathrm{M}-1)\psi_{\mathrm{M}-1}\left(\dfrac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)'+\psi_{\mathrm{M}-2}\left(\dfrac{\Phi_{s-2}}{\psi_\mathrm{M}}\right); \\
p=1, \qquad q=s-\mathrm{M}+1, \\
\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[\ m, 1\ ]\left(\dfrac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(m-1)}=\mathrm{M}\psi_\mathrm{M}\left(\dfrac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-1)}+(\mathrm{M}-1)\psi_{\mathrm{M}-1}\left(\dfrac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-2)}+\ldots\psi_1\left(\dfrac{\Phi_{s-\mathrm{M}-1}}{\psi_\mathrm{M}}\right); \\
p=0, \qquad q=s-\mathrm{M}, \\
\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[\ m, 0\ ]\left(\dfrac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(m)}=\psi_\mathrm{M}\left(\dfrac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(\mathrm{M})}+\psi_{\mathrm{M}-1}\left(\dfrac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-1)}+\ldots\psi_0\left(\dfrac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right);
\end{array}
\right\} \qquad (12.)
$$

the sum of all which will be found, on reference to the expressions for the formation of the Φs, to be equal to the first term of Φ_s, viz. $\phi_{\mathrm{N}-s}$; and consequently the coefficient of $\pi^{\mathrm{N}-s}$ vanishes for all values of s not exceeding the greatest value of q, viz. $\mathrm{N}-\mathrm{M}$. If, however, s is greater than $\mathrm{N}-\mathrm{M}$, by any number t, so that $s=\mathrm{N}-\mathrm{M}+t$, then the pairs of values

$$
\begin{array}{ll}
p=\mathrm{M}, & q=s, \\
p=\mathrm{M}-1, & q=s-1, \\
\quad . \ . & \quad . \ . \\
p=\mathrm{M}-t+1, & q=s-t+1
\end{array}
$$

are inadmissible, and the pairs

$$
\begin{array}{ll}
p=\mathrm{M}-t, & q=s-t, \\
p=\mathrm{M}-t-1, & q=s-t-1, \\
\quad . \ . & \quad . \ . \\
p=0, & q=s-\mathrm{M}
\end{array}
$$

alone remain; and consequently the coefficients of the powers of π, for $s>\mathrm{N}-\mathrm{M}$, do not vanish, and the remainder consists of a series of terms, the index of the highest power of π being

$$\mathrm{N}-\mathrm{M}+p-q=\mathrm{N}-s=\mathrm{N}-\mathrm{N}+\mathrm{M}-1=\mathrm{M}-1,$$

as it should be.

As an example, we may calculate by means of the formulæ given above, the final remainder in the external division of

$$\phi_4(\varrho)\pi^4+\phi_3(\varrho)\pi^3+\phi_2(\varrho)\pi^2+\phi_1(\varrho)\pi+\phi_0(\varrho)$$

by

$$\psi_3(\varrho)\pi^3+\psi_2(\varrho)\pi^2+\psi_1(\varrho)\pi+\psi_0(\varrho),$$

viz.

$$\Sigma_{n=0}^{n=4}\phi_n\pi^n-\Sigma_{m=0}^{m=2}\ \Sigma_{p=0}^{p=m}\ \Sigma_{q=0}^{q=2}\psi_m[m,p]\left(\frac{\Phi_q}{\psi_{\mathrm{M}}}\right)^{(m-p)}\pi^{m+p-q}.$$

The conditions

$$p=2,\ q=0 \text{ give } \Phi_0\pi^4$$

$$p=2,\ q=1 \text{ ——— } \Phi_1\pi^3$$

$$p=2,\ q=2 \text{ ——— } \Phi_2\pi^2$$

$$p=1,\ q=0 \text{ ——— } \left\{2\psi_2\left(\frac{\Phi_0}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^3$$

$$p=1,\ q=1 \text{ ——— } \left\{2\psi_2\left(\frac{\Phi}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi^2$$

$$p=1,\ q=2 \text{ ——— } \left\{2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi$$

$$p=0,\ q=0 \text{ ——— } \left\{\psi_2\left(\frac{\Phi_0}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^2$$

$$p=0,\ q=1 \text{ ——— } \left\{\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi$$

$$p=0,\ q=2 \text{ ——— } \left\{\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}.$$

Hence taking the sum of all the terms, the coefficients of π^4, π^3, π^2 vanish, and the final remainder is

$$\left\{\phi_1-2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)-\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi+\phi_0-\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_2}{\psi_2}\right).$$

These results may be compared with the actual division,

$$\psi_2\pi^2+\psi_1\pi+\psi_0\Big)\ \phi_4\pi^4+\phi_3\pi^3+\phi_2\pi^2+\phi_1\pi+\phi_0\left(\frac{\Phi_0}{\psi_2}\pi^2+\frac{\Phi_1}{\psi_2}\pi+\frac{\Phi_2}{\psi_2}\right.$$

$$\left.\begin{aligned}&\phi_4\pi^4+2\psi_2\left(\frac{\Phi_0}{\psi_2}\right)'\pi^3+\psi_2\left(\frac{\Phi_0}{\psi_2}\right)''\pi^2\\&\quad+\ \psi_1\left(\frac{\Phi_0}{\psi_2}\right)\pi^3+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)'\pi^2\\&\qquad\qquad+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\pi^2\end{aligned}\right\}$$

$$\Phi_1\pi^3+\left\{\phi_2-\psi_2\left(\frac{\Phi_0}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_0}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^2+\phi_1\pi+\phi_0$$

$$\Phi_1\pi^3+\quad\left.\begin{aligned}&2\psi^2\left(\frac{\Phi_1}{\psi_2}\right)'\pi^2+\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''\pi\\&+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\pi^2+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'\pi\\&\qquad\qquad+\psi_0\left(\frac{\Phi_1}{\psi_2}\right)\pi\end{aligned}\right\}$$

$$\begin{array}{ll}
\Phi_2\pi^2+\left\{\phi_1-\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi+\phi_0 & \\
\Phi_2\pi^2 & \left.\begin{array}{r} +2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'\pi+\psi_2\left(\frac{\Phi_2}{\psi_2}\right) \\ +\ \psi_1\left(\frac{\Phi_2}{\psi_2}\right)\pi+\psi_1\left(\frac{\Phi_2}{\psi_2}\right)' \\ +\psi_0\left(\frac{\Phi_2}{\psi_2}\right) \end{array}\right\} \\
\hline
\Phi_3\pi+\left\{\phi_0-\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_2}{\psi_2}\right)\right\}, &
\end{array}$$

which agrees with the results found above.

§ 5. *To divide $\Sigma_{n=0}^{n=N}\varrho^n\phi_n(\pi)$ internally by $\Sigma_{m=0}^{m=M}\varrho^m\psi_m(\pi)$.*

The first term of the quotient will be

$$\varrho^{N-M}\frac{\phi_N(\pi-M)}{\psi_M(\pi-M)}, \quad \ldots \ldots \quad (1.)$$

and the product of this into the divisor,

$$\Sigma_{m=0}^{m=M}\varrho^{N-M+m}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)}\phi_N(\pi-M+m). \quad \ldots \quad (2.)$$

The first term of the remainder will then be

$$\varrho^{N-1}\left\{\phi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi)}{\psi_M(\pi-1)}\phi_N(\pi-1)\right\}=\varrho^{N-1}\frac{1}{\psi_M(\pi-1)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi_{M-1}(\pi) \\ \phi_N(\pi-1) & \psi_M(\pi-1),\end{vmatrix} \quad \ldots \quad (3.)$$

and consequently the second term in the quotient will be

$$\left.\begin{array}{l}
\varrho^{N-M-1}\left\{\phi_{N-1}(\pi-M)-\frac{\psi_{M-1}(\pi-M)}{\psi_M(\pi-M-1)}\phi_N(\pi-M-1)\right\}\frac{1}{\psi_M(\pi-M)} \\
=\varrho^{N-M-1}\frac{1}{\psi_M(\pi-M)\psi_M(\pi-M-1)}\begin{vmatrix}\phi_{N-1}(\pi-M) & \psi_{M-1}(\pi-M) \\ \phi_N(\pi-M-1) & \psi_M(\pi-M-1)\end{vmatrix}
\end{array}\right\}. \quad \ldots \quad (4.)$$

The first term of the second remainder will then be

$$\left.\begin{array}{l}
\varrho^{N-2}\left\{\phi_{N-2}(\pi)-\frac{\psi_{M-1}(\pi)}{\psi_M(\pi-1)\psi_M(\pi-2)}\begin{vmatrix}\phi_{N-1}(\pi-1) & \psi_{M-1}(\pi-1) \\ \phi_N(\pi-2) & \psi_M\ (\pi-2)\end{vmatrix}\right\} \\
=\varrho^{N-2}\frac{1}{\psi_M(\pi-1)\psi_M(\pi-2)}\begin{vmatrix}\phi_{N-2}(\pi) & \psi_{M-1}(\pi) & 0 \\ \phi_{N-1}(\pi-1) & \psi_M(\pi-1) & \psi_{M-1}(\pi-1) \\ \phi_N(\pi-2) & 0 & \psi_M(\pi-2)\end{vmatrix}
\end{array}\right\}. \quad \ldots \quad (5.)$$

And it is not difficult to see that the first term of the rth remainder will be

$$\varrho^{N-r}\frac{1}{\psi_M(\pi-1)\psi_M(\pi-2)\ldots\psi_M(\pi-r)}\begin{vmatrix}\phi_{N-r}(\pi) & \psi_{M-1}(\pi) & \ldots & 0\\ \phi_{N-r+1}(\pi-1) & \psi_M\ (\pi-1) & \ldots & 0\\ \vdots & \vdots & & \vdots\\ \phi_N(\pi-r) & 0 & \ldots & \psi_M(\pi-r),\end{vmatrix}\quad\ldots\quad(6.)$$

in which determinant every column after the first consists of only two terms, viz. $\psi_{M-1}(\pi-s)$ and $\psi_M(\pi-s-1)$. Hence also the $(r+1)$th term of the quotient will be

$$\varrho^{N-M-r}\frac{1}{\psi_M(\pi-M)\psi_M(\pi-M-1)\ldots\psi_M(\pi-M-r)}\begin{vmatrix}\phi_{N-r}(\pi-M) & \psi_{M-1}(\pi) & \ldots & 0\\ \phi_{N-r+1}(\pi-M-1) & \psi_M(\pi-1) & \ldots & 0\\ \vdots & \vdots & & \vdots\\ \phi_N(\pi-r) & 0 & \ldots & \psi_M(\pi-r).\end{vmatrix}\quad(7.)$$

As to the other terms, than the first, of the various remainders. In the first remainder, the first term of which is given by (3.), the $(s+1)$th term will be found by making $n=N-s$, $m=M-s$, in the expression

$$\Sigma_{n=0}^{n=N}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)}\phi_N(\pi-M+m),$$

which gives

$$\varrho^{N-s}\left\{\phi_{N-s}(\pi)-\frac{\psi_{M-s}(\pi)}{\psi_M(\pi-s)}\phi_N(\pi-s)\right\}$$

$$=\varrho^{N-s}\frac{1}{\psi_M(\pi-s)}\begin{vmatrix}\phi_{N-s}(\pi) & \psi_{M-s}(\pi)\\ \phi_N(\pi-s) & \psi_M(\pi-s).\end{vmatrix}$$

Hence the entire first remainder may be expressed thus:

$$\Sigma_{s=0}^{s=M}\frac{\varrho^{N-s}}{\psi_M(\pi-s)}\begin{vmatrix}\phi_{N-s}(\pi) & \psi_{M-s}(\pi)\\ \phi_N(\pi-s) & \psi_M(\pi-s).\end{vmatrix}\quad\ldots\ldots\ldots\quad(8.)$$

Similarly, the general expression for the second remainder is

$$\Sigma_{n=0}^{n=N-1}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m-1}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)\psi_M(\pi-M+m-1)}\begin{vmatrix}\phi_{N-1}(\pi-M+m) & \psi_{M-1}(\pi-M+m)\\ \phi_N(\pi-M+m-1) & \psi_M\ (\pi-M+m-1),\end{vmatrix}$$

which may be transformed thus:

$$n=N-s,\quad m=M-s,$$

$$\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\left\{\phi_{N-s-1}(\pi)-\frac{\psi_{M-s}(\pi)}{\psi_M(\pi-s)\psi_M(\pi-s-1)}\begin{vmatrix}\phi_{N-1}(\pi-s) & \psi_{M-1}(\pi-s)\\ \phi_N(\pi-s-1) & \psi_M(\pi-s-1)\end{vmatrix}\right\}$$

$$=\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\frac{1}{\psi_M(\pi-s)\psi_M(\pi-s-1)}\begin{vmatrix}\phi_{N-s-1}(\pi\quad) & \psi_{M-s}(\pi\quad) & 0\\ \phi_{N-1}\ (\pi-s\quad) & \psi_M\ (\pi-s) & \psi_{M-1}(\pi-s\quad)\\ \phi_N\ (\pi-s-1) & 0 & \psi_M\ (\pi-s-1).\end{vmatrix}\quad(9.)$$

And generally the expression for the tth remainder may be written

$$\Sigma_{s=0}^{s=\mathrm{M}} \varrho^{\mathrm{N}-s-t+1} \frac{1}{\psi_{\mathrm{M}}(\pi-s)\psi_{\mathrm{M}}(\pi-s-1)\,..\,\psi_{\mathrm{M}}(\pi-s-t+1)} \begin{vmatrix} \phi_{\mathrm{N}-st+t}(\pi) & \psi_{\mathrm{M}-s}(\pi) & .. & 0 \\ \phi_{\mathrm{N}-t+1}(\pi-s) & \psi_{\mathrm{M}}(\pi-s) & .. & \psi_{\mathrm{M}-t+1}(\pi-s) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{\mathrm{N}}(\pi-s-t+1) & 0 & .. & \psi_{\mathrm{M}}(\pi-s-t+1). \end{vmatrix} \quad (10.)$$

The last remainder is the $(\mathrm{N}-\mathrm{M}+1)$th. Then

$$t=\mathrm{N}-\mathrm{M}+1,$$
$$\mathrm{N}-s-t+1=\mathrm{N}-s-\mathrm{N}+\mathrm{M}-1+1=\mathrm{M}-s$$
$$\mathrm{N}-t+1=\mathrm{N}\quad-\mathrm{N}+\mathrm{M}-1+1=\mathrm{M}$$
$$\mathrm{M}-t+1=\mathrm{M}\quad-\mathrm{N}+\mathrm{M}-1+1=2\mathrm{M}-\mathrm{N},$$

and the remainder in question

$$=\Sigma_{s=0}^{s=\mathrm{M}} \varrho^{\mathrm{M}-s} \frac{1}{\psi_{\mathrm{M}}(\pi-s)\psi_{\mathrm{M}}(\pi-s-1)\,..\,\psi_{\mathrm{M}}(\pi-s-\mathrm{N}+\mathrm{M})} \quad . \quad . \quad . \quad . \quad (11.)$$
$$\times \begin{vmatrix} \phi_{\mathrm{M}-s}(\pi) & \psi_{\mathrm{M}-s}(\pi) & .. & 0 \\ \phi_{\mathrm{M}}(\pi-s) & \psi_{\mathrm{M}}(\pi-s) & .. & \psi_{2\mathrm{M}-\mathrm{N}}(\pi-s) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{\mathrm{N}}(\pi-s-\mathrm{N}+\mathrm{M}) & 0 & .. & \psi_{\mathrm{M}}(\pi-s-\mathrm{N}+\mathrm{M}), \end{vmatrix}$$

in which the coefficient of ϱ^{M} vanishes, as it should. The last term, viz. that independent of ϱ,

$$=\frac{1}{\psi_{\mathrm{M}}(\pi-\mathrm{M})\psi_{\mathrm{M}}(\pi-\mathrm{M}-1)\,..\,\psi_{\mathrm{M}}(\pi-\mathrm{N})} \quad . \quad . \quad . \quad . \quad (12.)$$
$$\times \begin{vmatrix} \phi_0(\pi) & \psi_0(\pi) & .. & 0 \\ \phi_{\mathrm{M}}(\pi-\mathrm{M}) & \psi_{\mathrm{M}}(\pi-\mathrm{M}) & .. & \psi_{2\mathrm{M}-\mathrm{N}}(\pi-\mathrm{M}) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{\mathrm{N}}(\pi-\mathrm{N}) & 0 & .. & \psi_{\mathrm{M}}(\pi-\mathrm{N}); \end{vmatrix}$$

and if $\mathrm{N}=1$, the result agrees with that given by Mr. RUSSELL*.

§ 6. *To divide* $\Sigma_{n=0}^{n=\mathrm{N}}\varrho^n\phi_n(\pi)$ *externally by* $\Sigma_{m=0}^{m=\mathrm{M}}\varrho^m\psi_m(\pi)$.

The first term of the quotient will be

$$\varrho^{\mathrm{N}-\mathrm{M}} \frac{\phi_{\mathrm{N}}(\pi)}{\psi_{\mathrm{M}}(\pi+\mathrm{N}-\mathrm{M})}. \quad . \quad . \quad . \quad . \quad (1.)$$

The first remainder

$$\left.\begin{aligned} &\Sigma_{n=0}^{n=\mathrm{N}}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=\mathrm{M}}\varrho^{\mathrm{N}-\mathrm{M}+m}\frac{\psi_m(\pi+\mathrm{N}-\mathrm{M})}{\psi_{\mathrm{M}}(\pi+\mathrm{N}-\mathrm{M})}\phi_{\mathrm{N}}(\pi) \\ &=\Sigma_{s=0}^{s=\mathrm{M}}\varrho^{\mathrm{N}-s}\left\{\phi_{\mathrm{N}-s}(\pi)-\frac{\psi_{\mathrm{M}-s}(\pi+\mathrm{N}-\mathrm{M})}{\psi_{\mathrm{M}}(\pi+\mathrm{N}-\mathrm{M})}\phi_{\mathrm{N}}(\pi)\right\}, \end{aligned}\right\} \quad . \quad . \quad . \quad (2.)$$

* Philosophical Transactions, vol. cli. p. 72.

the first term of which is

$$\varrho^{N-1}\left\{\phi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi+N-M)}{\psi_M(\pi+N-M)}\,\phi_N(\pi)\right\},$$

whence the second term of the quotient will be

$$\left.\begin{aligned}&\varrho^{N-M-1}\frac{1}{\psi_M(\pi+N-M-1)}\left\{\phi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi+N-M)}{\psi_M(\pi+N-M)}\,\phi_N(\pi)\right\}\\ &\quad=\varrho^{N-M-1}\frac{1}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi^{M-1}\ \pi+N-M) \\ \phi_N\ (\pi) & \psi_M\ (\pi+N-M).\end{vmatrix}\end{aligned}\right\}\quad . \quad . \quad (3.)$$

Similarly, the second remainder will be

$$\left.\begin{aligned}&\Sigma_{n=0}^{n=N-1}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m-1}\frac{\psi_m(\pi+N-M-1)}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi_{M-1}(\pi+N-M) \\ \phi_N(\pi) & \psi_M\ (\pi+N-M)\end{vmatrix}\\ &=\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\left\{\phi_{N-s-1}(\pi)-\frac{\psi_{M-s}(\pi+N-M-1)}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi_{M-1}(\pi+N-M) \\ \phi_N(\pi) & \psi_M(\pi+N-M)\end{vmatrix}\right\}\\ &=\Sigma_{s=0}^{s=m}\varrho^{N-s-1}\frac{1}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-s-1}(\pi) & \psi_{M-s}(\pi+N-M-1) & 0 \\ \phi_{N-1}\ (\pi) & \psi_M\ (\pi+N-M-1) & \psi_{M-1}(\pi+N-M) \\ \phi_N\ (\pi) & 0 & \psi_M\ (\pi+N-M)\end{vmatrix}\end{aligned}\right\}(4.)$$

The tth remainder

$$\left.\begin{aligned}&=\Sigma_{s=0}^{s=M}\varrho^{N-s-t+1}\frac{1}{\psi_M(\pi+N-M-t+1)\psi_M(\pi+N-M-t+2)\,..\,\psi_M(\pi+N-M)}\\ &\qquad\times\begin{vmatrix}\phi_{N-s-t+1}(\pi) & \psi_{M-s}(\pi+N-M-t+1)\ .. & 0 \\ \phi_{N-t+1}(\pi) & \psi_M(\pi+N-M-t+1)\ .. & \psi_{M-t+1}(\pi+N-M) \\ \vdots & \vdots & \vdots \\ \phi_N(\pi) & 0\ \ . & \psi_M(\pi+N-M)\end{vmatrix};\end{aligned}\right\}\quad . \quad . \quad . \quad (5.)$$

and the last remainder, viz. the $(N-M+1)$th,

$$=\Sigma_{s=0}^{s=M}\varrho^{M-s}\frac{1}{\psi_M(\pi)\psi_M(\pi+1)\,..\,\psi_M(\pi+N-M)}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

$$\times\begin{vmatrix}\phi_{M-s}(\pi) & \psi_{M-s}(\pi)\ .. & 0 \\ \phi_M(\pi) & \psi_M(\pi)\ .. & \psi_{2M-N}(\pi+N-M) \\ \vdots & \vdots & \vdots \\ \phi_N(\pi) & 0\ .. & \psi_M(\pi+N-M).\end{vmatrix}$$

In the case considered by Mr. RUSSELL, viz. $M=1$, (6.) gives only the single term

$$\{\psi_1(\pi)\psi_1(\pi+1)\,..\,\psi_1(\pi+N-1)\}^{-1}\begin{vmatrix}\phi_0(\pi) & \psi_0(\pi)\ .. & 0 \\ \phi_1(\pi) & \psi_1(\pi)\ .. & 0 \\ \vdots & \vdots & \vdots \\ \phi_{N-1}(\pi) & 0\ .. & \psi_0(\pi+N-1) \\ \phi_N(\pi) & 0\ .. & \psi_1(\pi+N-1),\end{vmatrix}$$

and in this the coefficient of $\phi_i(\pi)$ is $(-)^i\times$

$$\begin{vmatrix} \psi_0(\pi) & 0 & .. & 0 & 0 & .. & 0 & 0 \\ \psi_1(\pi) & \psi_0(\pi+1) & .. & 0 & 0 & .. & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & .. & \psi_0(\pi+i-1) & 0 & .. & 0 & 0 \\ 0 & 0 & .. & 0 & \psi_1(\pi+i) & .. & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & .. & 0 & 0 & .. & \psi_1(\pi+N-2) & \psi_0(\pi+N-1) \\ 0 & 0 & .. & 0 & 0 & .. & 0 & \psi_1(\pi+N-1) \end{vmatrix}$$

$$=\psi_0(\pi)\psi_0(\pi+1)\,..\,\psi_0(\pi+i-1)\psi_1(\pi+i)\,..\,\psi_1(\pi+N-2)\psi_1(\pi+N-1);$$

whence the whole expression

$$=\Sigma_{i=0}^{i=N}(-)^i\phi_i(\pi)\frac{\psi_0(\pi)\psi_0(\pi+1)\,..\,\psi_0(\pi+i-1)}{\psi_1(\pi)\psi_1(\pi+1)\,..\,\psi_1(\pi+i-1)},$$

which agrees with the result given in the Philosophical Transactions, vol. cli. p. 73.

In the particular case of $N=4$, $M=2$, the final remainder in internal division is

$$\varrho\frac{1}{\psi_2(\pi-1)\psi_2(\pi-2)\psi_2(\pi-3)}\begin{vmatrix} \phi_1(\pi) & \psi_1(\pi) & 0 & 0 \\ \phi_2(\pi-1) & \psi_2(\pi-1) & \psi_1(\pi-1) & \psi_0(\pi-1) \\ \phi_3(\pi-2) & 0 & \psi_2(\pi-2) & \psi_1(\pi-2) \\ \phi_4(\pi-3) & 0 & 0 & \psi_2(\pi-3) \end{vmatrix}$$

$$+\frac{1}{\psi_2(\pi-2)\psi_2(\pi-3)\psi_2(\pi-4)}\begin{vmatrix} \phi_0(\pi) & \psi_0(\pi) & 0 & 0 \\ \phi_2(\pi-2) & \psi_2(\pi-2) & \psi_1(\pi-2) & \psi_0(\pi-2) \\ \phi_3(\pi-3) & 0 & \psi_2(\pi-3) & \psi_1(\pi-3) \\ \phi_4(\pi-4) & 0 & 0 & \psi_2(\pi-4) \end{vmatrix};$$

and in external division it is

$$\varrho\frac{1}{\psi_2(\pi)\psi_2(\pi+1)\psi_2(\pi+2)}\begin{vmatrix} \phi_1(\pi) & \psi_1(\pi) & 0 & 0 \\ \phi_2(\pi) & \psi_2(\pi) & \psi_1(\pi+1) & \psi_0(\pi+2) \\ \phi_3(\pi) & 0 & \psi_2(\pi+1) & \psi_1(\pi+2) \\ \phi_4(\pi) & 0 & 0 & \psi_2(\pi+2) \end{vmatrix}$$

$$+\frac{1}{\psi_2(\pi)\psi_2(\pi+1)\psi_2(\pi+2)}\begin{vmatrix} \phi_0(\pi) & \psi_0(\pi) & 0 & 0 \\ \phi_2(\pi) & \psi_2(\pi) & \psi_1(\pi+1) & \psi_0(\pi+2) \\ \phi_3(\pi) & 0 & \psi_2(\pi+1) & \psi_1(\pi+1) \\ \phi_4(\pi) & 0 & 0 & \psi_2(\pi) \end{vmatrix}.$$

The expressions (10.) for the formation of the Φs admit of further development thus:

$$\Phi_0=\phi_N$$

$$\Phi_1=\phi_{N-1}-M\psi_M\left(\frac{\phi_N}{\psi_M}\right)'-\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)$$

$$\begin{aligned}\Phi_2=\phi_{N-2}-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''-(M-1)\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'-\psi_{M-2}\left(\frac{\phi_N}{\psi_M}\right)\\ +\quad M^2\psi_M\left(\frac{\phi_N}{\psi_M}\right)''+M\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'+M\psi_M\left(\frac{\psi_{N-1}}{\psi_M}\right)'\left(\frac{\phi_N}{\psi_M}\right)-M\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'\\ +M\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'+\psi_{M-1}\left(\frac{\psi_{N-1}}{\psi_M}\right)\left(\frac{\phi_N}{\psi_M}\right)-\psi_{M-1}\left(\frac{\phi_{N-1}}{\psi_M}\right)\end{aligned}$$

$$\begin{aligned}=\frac{(M+1)M}{1.2}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''+(M+1)\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'+\left\{-\psi_{M-2}+M\psi_M\left(\frac{\psi_{M-1}}{\psi_M}\right)'+\psi_{M-1}\left(\frac{\psi_{N-1}}{\psi_M}\right)\right\}\left(\frac{\phi_N}{\psi_M}\right)\\ -M\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\phi_{N-1}}{\psi_M}\right)\\ +\qquad\phi_{N-2};\end{aligned}$$

or writing, by analogy to the Φs,

$$\Psi_0=\psi_{M-1}$$

$$-\Psi_1=\psi_{M-2}-M\psi_M\left(\frac{\Psi_0}{\psi_M}\right)'-\psi_{M-1}\left(\frac{\Psi_0}{\psi_M}\right),$$

the expression for Φ_2 becomes

$$\begin{aligned}\Phi_2=\frac{(M+1)M}{1.2}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''+(M+1)\Psi_0\left(\frac{\phi_N}{\psi_M}\right)'+\Psi_1\left(\frac{\phi_N}{\psi_M}\right)\\ -M\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'-\qquad\Psi_0\left(\frac{\phi_{N-1}}{\psi_M}\right)\\ +\quad\psi_M\left(\frac{\phi_{N-2}}{\psi_M}\right).\end{aligned}$$

And so likewise writing

$$\begin{aligned}\Psi_2=\psi_{M-3}-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Psi_0}{\psi_M}\right)''-(M-1)\psi_{M-1}\left(\frac{\Psi_0}{\psi_M}\right)'-\psi_{M-2}\left(\frac{\Psi_0}{\psi_M}\right)\\ -M\psi_M\left(\frac{\Psi_1}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\Psi_1}{\psi_M}\right),\end{aligned}$$

it will be found that

$$\begin{aligned}\Phi_3={}_{N-3}-\frac{M(M-1)(M-2)}{1.2.3}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)'''-\frac{(M-1)(M-2)}{1.2}\psi_{M-1}\left(\frac{\Phi_0}{\psi_M}\right)''-(M-2)\psi_{M-2}\left(\frac{\Phi_0}{\psi_M}\right)'-\psi_{M-3}\left(\frac{\Phi_0}{\psi_M}\right)\\ -\qquad\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_1}{\psi_M}\right)''-\qquad(M-1)\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)'-\qquad\psi_{M-2}\left(\frac{\Phi_1}{\psi_M}\right)\\ -\qquad M\psi_M\left(\frac{\Phi_2}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\Phi_2}{\psi_M}\right)\end{aligned}$$

$$
\begin{aligned}
=-\frac{(M+2)(M+1)M}{1.2.3}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''' &- \frac{(M+2)(M+1)}{1.2}\Psi_0\left(\frac{\phi_N}{\psi_M}\right)'' - (M+2)\Psi_1\left(\frac{\phi_N}{\psi_M}\right)' - \Psi_2\left(\frac{\phi_N}{\psi_M}\right) \\
+ \quad \frac{(M+1)M}{1.2}\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'' &+ \quad (M+1)\Psi_0\left(\frac{\phi_{N-1}}{\psi_M}\right)' + \quad \Psi_1\left(\frac{\phi_{N-1}}{\psi_M}\right) \\
- \quad M\psi_M\left(\frac{\phi_{N-2}}{\psi_M}\right)'' &- \quad \Psi_0\left(\frac{\phi_{N-2}}{\psi_M}\right) \\
+ \quad \psi_M\left(\frac{\phi_{N-3}}{\psi_M}\right). &
\end{aligned}
$$

But the law of the expressions in the first form having been established above, it is unnecessary to pursue these latter formulæ further.